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ILLUMINANCE DURING A SOLAR ECLIPSE WITH LIMB DARKENING: A MATHEMATICAL MODEL

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ABSTRACT

We present a mathematical model that predicts the variation of illuminance during a solar eclipse, considering continuous effects of limb darkening. We assume that (1) the Sun and the Moon constitute perfect spheres, (2) the Moon crosses the Sun with a constant apparent velocity, and (3) sunspots, prominences, and coronae can be neglected. We compare predictions of this model with actual measurements made by Möllmann & Vollmer (2006) during a total solar eclipse in Turkey, and with predictions of existing models. The new model is shown to describe the actual phenomenon more accurately than existing models.

Key words : solar eclipse — illuminance — limb darkening

1. INTRODUCTION

A solar eclipse is one of the most notable phenomena exhibited in the sky. Since ancient times, the Sun hidden by the Moon has captured people's attention, and thus, analysis of various results of a solar eclipse is the topic of high interest. Many studies have been made to predict and measure the variations of solar properties such as air and surface temperatures, or the color of the Sun (see Gedzelman 1975 or Littmann et al. 2008). Among these we focus on illuminance, about which surprisingly few works have been made, as mentioned in Möllmann & Vollmer (2006).

Solar illuminance, defined by the solar radiation energy received by a unit area of terrestrial surface, indicates how bright the Sun is in presence of no other significant light sources. The illuminance is proportional to the surface integral of the light intensity of the light source. Although many other definitions are possible, we refer here to light intensity as the radiation energy originating from a unit solid angle of the light source. The central part of the Sun exhibits a higher light intensity than its outer part, and this phenomenon is referred to limb darkening.

Möllmann & Vollmer (2006) constructed a model to predict the illuminance during a solar eclipse. In their model, it was assumed that the Sun exhibits a uniform light intensity throughout its surface, and therefore the illuminance during a solar eclipse was assumed to be proportional to the visible, i.e., not covered by the Moon, area of the Sun. They also made a series of measurements of the illuminance during a total eclipse in Southern Turkey, and compared the model with the measurements to confirm the validity of the model.

A few years later, Vollmer (2009) improved this model by taking into account the effect of limb darkening in a discrete way. That is, he assigned different light intensity values to two distinct parts of the Sun, i.e., the area closer to the center than 0.6 times of the solar radius, and the area farther than the limit.

Although the two models perform well in fitting the actual measurement of illuminance presented by Möllmann & Vollmer (2006), they failed to describe the continuous effects of limb darkening. In this work, after making some assumptions, we analyze quantitatively the effects of limb darkening in Section 2, while in Section 3 we construct a new model which includes the desired improvements. In Section 4, we compare statistically our model and those of Möllmann & Vollmer (2006) and Vollmer (2009) with the actual measurement of illuminance presented by Möllmann & Vollmer (2006), in order to determine their accuracy in predicting the illuminance during a solar eclipse.

2. PRELIMINARIES

2.1 Assumptions

To simplify the modeling process while preserving its accuracy, following assumptions were made in prior to constructing a model for solar eclipse.

- The Sun and the Moon are perfect spheres.

- The apparent motion of two bodies can be considered as a motion of two disks with a constant relative velocity.

- The angular radii of two bodies remain constant during the eclipse.

- Effects of Sunspots, prominences, and coronae are

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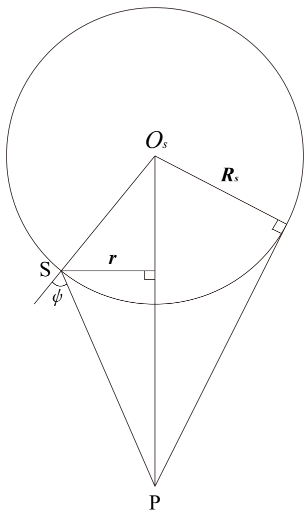

Fig. 1.— A diagram of the Sun, seen from a distant point.

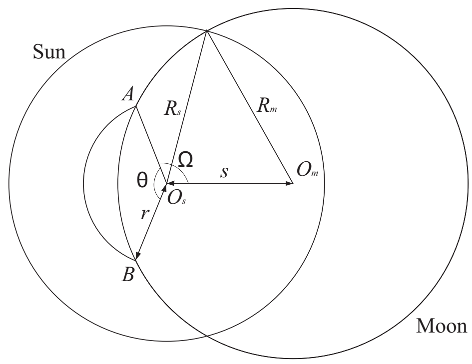

Fig. 2.— Geometry for the case of the solar center covered by the Moon.

neglected.

- An annular eclipse, i.e., the complete inclusion of the Moon in the Sun is not considered.

2.2 Limb Darkening

The light intensity of the Sun is highest at its center, and decreases with increasing distance from the center. This phenomenon is known as limb darkening. The light intensity at a point on the source S observed from a distant point P, as illustrated in Fig. 1., is related to $\cos\psi$ as follows, (see Cox 2001)

$$\frac{I(\psi)}{I(0)} = a_0 + a_1 \cos\psi + a_2 \cos^2\psi, \quad (1)$$

where $I(0)$ is the central light intensity and the coefficients a_k's satisfy $a_0 + a_1 + a_2 = 1$. In case of the solar radiation at wavelength 5500Å, the coefficients are given by

$$a_0 = 0.3, \quad a_1 = 0.93, \quad a_2 = -0.23. \quad (2)$$

We use these formulae when constructing our model in the following section.

If P is a terrestrial point, the distance from P to the Sun is large enough compared to the solar radius R_s to approximate ψ by the angle SO_sP, or $\angle SO_sP$. The apparent distance of S from the solar center, r, is then given by $r = R_s \sin\angle SO_sP$, and thus the light intensity $I(r)$ at a point with an apparent distance from the solar center r, is generally expressed as,

$$\begin{aligned}\frac{I(r)}{I(0)} &= 0.3 + 0.93\cos\psi - 0.23\cos^2\psi \\ &= 0.07 + 0.93\sqrt{1-\left(\frac{r}{R_s}\right)^2} + 0.23\left(\frac{r}{R_s}\right)^2. \quad (3)\end{aligned}$$

We recall that solar illuminance is proportional to the surface integral of the light intensity of the light source. Using radial symmetry, the entire illuminance of the Sun is given by

$$L_{max} = \int_0^{R_s} I(r) 2\pi r dr = 0.805\pi I(0) R_s^2 = 0.805 L_0, \quad (4)$$

where $L_0 = I(0)\pi R_s^2$ indicates the illuminance assuming the central light intensity to be constant over the solar surface. This implies that limb darkening has a significant effect on the solar illuminance.

3. A MODEL FOR THE SOLAR ILLUMINANCE

In this section we evaluate the solar illuminance when the Moon covers the Sun, either partially or completely. We use s, the distance between centers of two celestial objects, as a parameter.

3.1 Solar Center Covered

First, we consider the case in which the Moon covers the solar center but does not cover the Sun completely (see the geometry in Fig. 2). For this condition, we require $R_m - R_s < s \le R_m$ when $R_m > R_s$, and $0 < s \le R_m$ when $R_m \le R_s$.

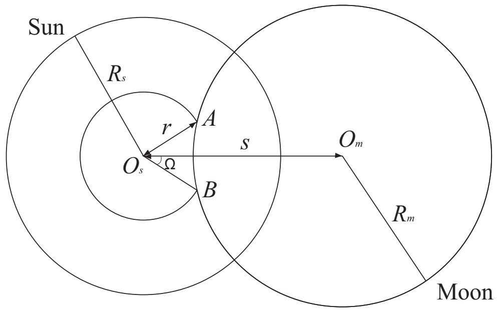

Fig. 3.— Geometry for the case of the solar center not covered by the Moon.

Applying the law of cosines on triangle O_sO_mA yields

$$\cos\Omega = \frac{s^2 + r^2 - R_m^2}{2sr}, \tag{5}$$

and thus

$$\theta = 2\pi - 2\Omega = 2\pi - 2\arccos\frac{s^2 + r^2 - R_m^2}{2sr}. \tag{6}$$

In order to obtain the global illuminance, we integrate the illuminance over infinitesimally thin arcs AB of radius r and central angle θ:

$$L(s) = \int_{R_m - s}^{R_s} I(r) r\theta dr. \tag{7}$$

Substitution of (3) and (6) into (7) gives the solar illuminance in terms of the central intensity $I(0)$.

3.2 Solar Center Uncovered

Next, we consider the case when the Moon covers the Sun only partially, so that it does not cover the solar center, i.e., $R_m < s \leq R_m + R_s$. As can be seen from Fig. 3., (5) still holds, and so does (6). However, the illuminance formula (7) requires some modifications.

For arcs with radius smaller than $s - R_m$, the central angle equals 2π, not θ. Therefore in this case, (7) becomes

$$L(s) = \int_0^{s-R_m} I(r)2\pi r dr + \int_{s-R_m}^{R_s} I(r) r\theta dr. \tag{8}$$

Again, substitution of (3) and (6) into (8) gives the solar illuminance in terms of the central intensity $I(0)$.

3.3 Illuminance Expressed with the Parameter s and t

It can be taken for granted that when $0 \leq s < R_m - R_s$, which is only possible if $R_m > R_s$, the Moon covers the Sun completely to block any illuminance. In this case, one gets

$$L(s) = 0. \tag{9}$$

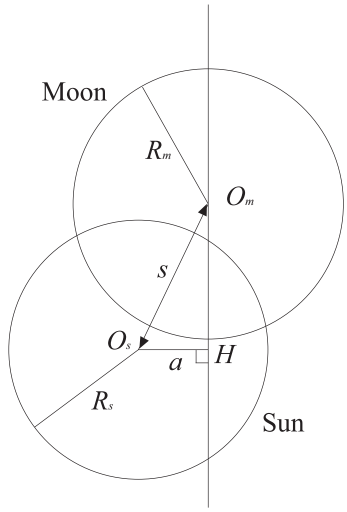

Fig. 4.— Movement of the Moon over the Sun.

To summarize, the illuminance $L(s)$ can be expressed with the parameter s as follows.

$$\begin{cases} (7) & \max(0, R_m - R_s) < s \leq R_m \\ (8) & R_m < s \leq R_m + R_s \\ (9) & 0 \leq s \leq \max(0, R_m - R_s) \end{cases} \tag{10}$$

Now, we investigate how the parameter s is related to the time t, so that the illuminance becomes a function of time. The solar eclipse can be divided into three time intervals; during the first one, the Moon starts to cover the Sun; during the second one, the Sun is completely hidden behind the Moon; and finally during the third one, the Moon gradually uncovers the Sun. The second interval, or the total eclipse, occurs when $0 \leq s \leq \max(0, R_m - R_s)$. The others, or the partial eclipse, occurs when $\max(0, R_m - R_s) < s \leq R_m + R_s$. Whether the total eclipse occurs or not depends on the solar and Lunar radii, and the path of the Moon.

Let $2T$ indicate the total duration of an eclipse, and t the time since the beginning of the eclipse. By definition, $s = R_m + R_s$ at $t = 0$ and at $t = 2T$. If H is the closest point on the path of the Lunar center to the solar center, at time $t = 0$, $O_mH(0) = \sqrt{(R_m + R_s)^2 - a^2}$ (see Fig. 4). Since the Lunar center must move a distance of $2\sqrt{(R_m + R_s)^2 - a^2}$ during the time $2T$ with constant velocity, $O_mH(t)$ is a function of time as follows:

$$O_mH(t) = \mid 1 - \frac{t}{T} \mid \sqrt{(R_m + R_s)^2 - a^2}. \tag{11}$$

This implies that s, as a function of time, is given by:

$$\begin{aligned} s(t) &= \sqrt{a^2 + O_mH^2} \\ &= \sqrt{(1 - \frac{t}{T})^2 (R_m + R_s)^2 + (\frac{2t}{T} - \frac{t^2}{T^2}) a^2}. \end{aligned} \tag{12}$$

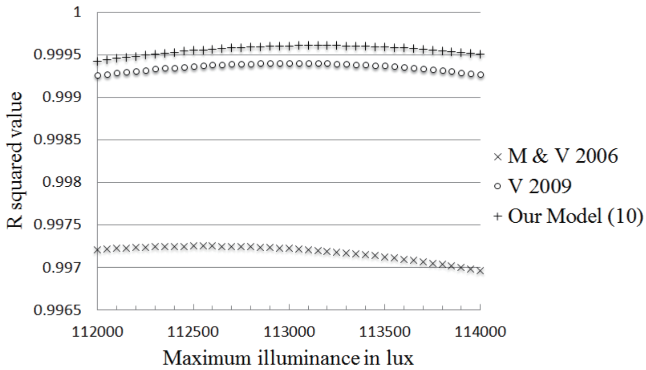

Fig. 5.— R^2 values as a function of L_{max}.

Substitution of (3), (6) and (12) into (10) finally yields the solar illuminance as a function of time. The exact evaluation of this relation requires some numerical integrations, which we perform using the Mathematica software.

4. COMPARISON

In this section, we compare the illuminance predicted by the new model in Section 3 with an actual measurement, and with other models presented by Möllmann & Vollmer (2006) and Vollmer (2009). The solar illuminance was measured by Möllmann and Vollmer (2006) in March 29th, 2006, during a total solar eclipse in southern Turkey. Note that there were no eruptive phenomena or significant features on the apparent solar disk over the illuminance measurement period, except for two small active regions on the east limb (see http://www.solarmonitor.org/).

The instrumental absolute accuracy (2.5%) and various errors, including the inclination and read-out errors, are not as big as few percents (see Möllmann & Vollmer 2006 for details of the eclipse observation). The measured data are displayed in Table 1. The time indication refers to the time in seconds since the beginning of the eclipse, and the illuminance indicates the solar illuminance in lux at the corresponding time.

The duration of the eclipse was $2T = 9406s$ including $t_{tot} = 224s$ of totality. We assume that this is a perfect total eclipse, i.e., $a = 0$. Then s is a linear function of t given by

$$s = O_m H(t) = \mid 1 - \frac{t}{T} \mid (R_m + R_s). \qquad (13)$$

The conditions on s for a total or partial eclipse imply that when s is between 0 and $R_m - R_s$ the totality occurs, and when s is between $R_m - R_s$ and $R_m + R_s$ the partiality occurs. Since the partiality duration $t_{par} = 2T - t_{tot}$ is 9182s, the rate of their lengths

$$\frac{t_{tot}}{t_{par}} = \frac{R_m - R_s}{2R_s} = 0.0244 \qquad (14)$$

Table 1.
Solar illuminance during a total eclipse in 2006, from Möllmann & Vollmer (2006).

Time (s)	Illuminance (lux)	Time (s)	Illuminance (lux)
30	111 800	4880	870
390	109 900	4890	1030
750	104 700	4900	1261
930	101 400	4910	1432
1110	98 100	4920	1608
1290	94 100	4930	1815
1470	89 700	4940	1973
1650	85 100	4950	2200
1830	80 000	4960	2460
2010	76 100	4970	2650
2190	70 700	4980	2850
2370	65 300	4990	3060
2550	59 200	5000	3270
2730	53 900	5010	3500
2910	47 600	5020	3700
3090	42 000	5030	3980
3270	36 900	5040	4380
3450	30 800	5050	4600
3570	26 800	5060	4840
3630	24 700	5250	7780
3810	19 200	5310	9410
3990	13 900	5370	12 350
4170	9300	5430	13 700
4290	6200	5610	19 180
4350	4600	5910	29 600
4410	3400	6090	34 700
4470	2200	6270	42 000
4530	1400	6570	52 400
4598	5	6690	56 200
4710	5	6990	63 500
4810	6	7170	70 500
4820	50	7590	82 300
4825	116	7770	87 500
4835	228	7950	92 600
4840	343	8130	96 400
4850	447	8970	112 500
4860	570	9390	113 000
4870	719		

Table 2.
Optimal R^2 values of the three models for illuminance described in the text.

Model	L_{max}	R^2
Möllmann & Vollmer (2006)	112 557 lx	0.9973
Vollmer (2009)	113 015 lx	0.9994
Our model (10)	113 147 lx	0.9996

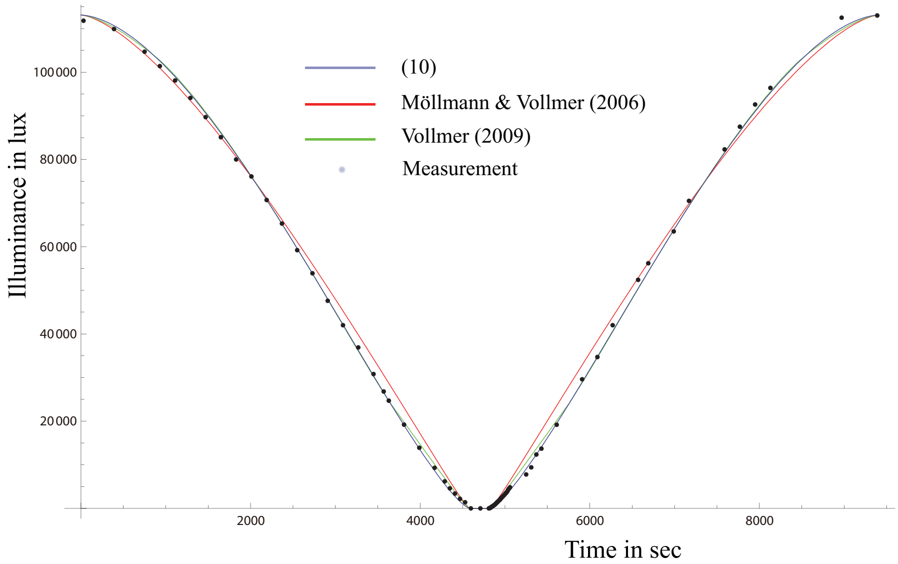

Fig. 6.— Comparison between our model (10) and those of Möllmann & Vollmer (2006) and Vollmer (2009).

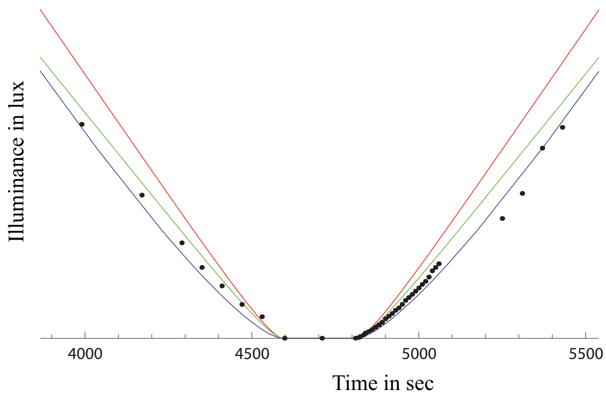

Fig. 7.— Enlargement of the central area of Fig. 6.

gives the ratio of the two angular radii, that is,

$$\frac{R_m}{R_s} = 1.0488. \tag{15}$$

Using all the previous information, the various models predict the illuminance with time as a multiple of a constant L_{max}, which indicates the maximum illuminance exhibited without any effects of eclipse. For a quantitative comparison, it is necessary to specify L_{max} and make a prediction in an absolute scale, rather than in a relative one. However, L_{max} has not been measured and it can only be inferred observationally to be between 112,000 and 114,000 lx, most likely about 113,000 lx. Therefore, we set L_{max} as a parameter varying between 112,000 and 114,000 lx, and make a quantitative comparison between the various models and the measurement.

A quantitative comparison can be made using the coefficient of determination, often referred to as the R^2 value. Coefficient of determination, defined as

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - y_i')^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} \tag{16}$$

indicates how closely a series of data y_i''s mimic an actual series of data y_i's. In (16), $\bar{y}$ indicates the mean of the y_i's. The closer the coefficient is to unity, the better the two datasets are in agreement (see Devore & Berk 2011).

For L_{max} varying by 50 lx ranging from 112,000 to 114,000 lx, our model (10) and the other two considered here (Möllmann & Vollmer 2006; Vollmer 2009) each predicts a series of values for the illuminance, to be compared with the measured ones shown in Table 1. The coefficients of determination for these three series of data are drawn in Fig. 5. with L_{max}. It shows that for all practical values of the parameter L_{max}, the model (10) fits the measurement better than the existing models from Möllmann & Vollmer (2006) where the limb darkening was neglected, and Vollmer (2009) where the limb darkening was considered discretely, although all the three models exhibit a decent level of resemblance.

In Table 2, the L_{max} values that give the highest R^2 value, and the corresponding R^2 values for each model are presented. For the case of L_{max} =113,100 lx, predictions of the models and the measurement are plotted in Fig. 6 and 7. magnifies the central part, i.e., $3900s \leq t \leq 5506s$ of Fig. 6.

5. CONCLUSION

Considering the continuous effects of limb darkening, we constructed a model for the variation of illuminance during a solar eclipse. The accuracy of the model is improved when compared with preceded models in which the effects of limb darkening is either neglected or only considered discretely. The improvement is especially distinguishable in vicinity of $t = 4591s$ and $t = 4815s$ where the switch between total and partial eclipse occurs. The coefficient of determination which indicates the similarity between the model prediction and the actual measurement equals 0.9996, implying that the model describes the phenomenon very accurately.

In this study, we assumed monochromatic light of wavelength 5500Å for the limb darkening. The application of an actual wavelength distribution for the solar illuminance may improve the accuracy of the model further.

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